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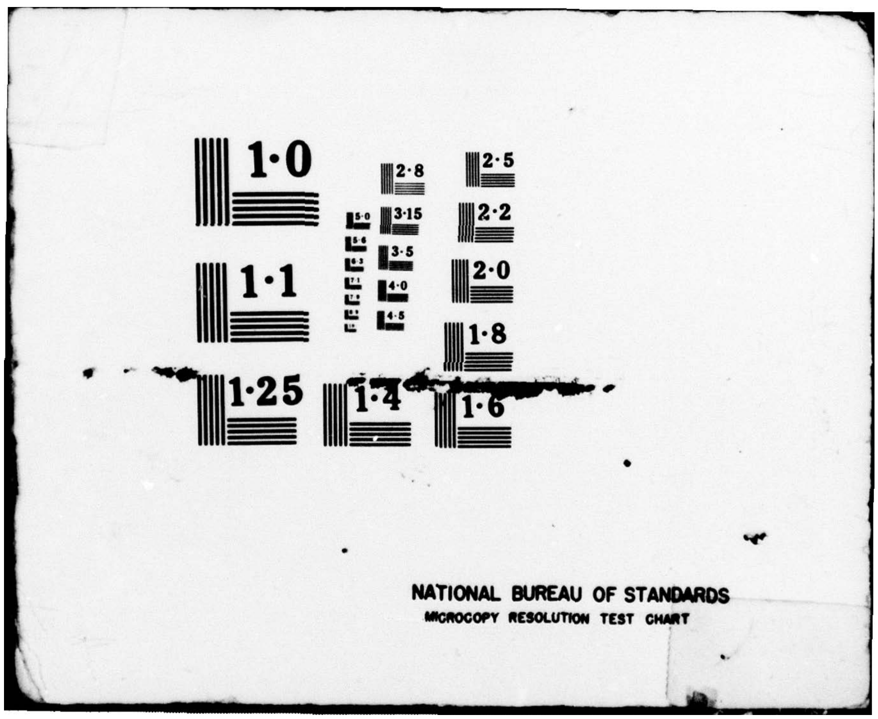
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THE EQUATION OF A GEODESIC LINE ON THE SURFACE OF AN OBLATE ELLIPSOID OF REVOLUTION

by

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The Equation of a Geodesic Line on the Surface of an Oblate Ellipsoid of Revolution

Jan Panasiuk

Let us consider the surface of an oblate ellipsoid of revolution in the form:

$$\vec{r} = [a\cos u\cos(\lambda-\lambda_G),\ a\cos u\sin(\lambda-\lambda_G),\ b\sin u],$$
$$(u,\lambda)\in\omega = \left\{(u,\lambda):\ u\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle;\ \lambda\in\langle -\pi,\pi)\right\}. \tag{1}$$

a, b – semi-axes of the meridian section $\lambda = \lambda_G$,

λ_G – established value of parameter λ.

The metric of a line on the surface is dependent on the quantities:

$$\begin{aligned} E &= |\vec{r}_u|^2 = \left(a\sqrt{1-e^2\cos^2 u}\right)^2,\\ F &= \vec{r}_u\cdot\vec{r}_\lambda = 0,\\ G &= |\vec{r}_\lambda|^2 = (a\cos u)^2,\\ H &= |\vec{r}_u x \vec{r}_\lambda| = a^2\cos u\sqrt{1-e^2\cos^2 u}. \end{aligned} \tag{2}$$

The first-order differential equation for a geodesic line placed on the surface (1) has the form:

$$a\cos u\sin A = \text{const} = c. \tag{3}$$

Parameter A in (3) designates the direction angle:

$$A = \measuredangle(\vec{r}_u, d\vec{r}) = \operatorname{arc\,ctg}\left(\frac{E\dfrac{du}{d\lambda}+F}{H}\right). \tag{4}$$

Symbol c – a certain arbitrary constant taking values from the interval $\langle -a, a\rangle$. Including (2) in (4), we have:

$$\operatorname{ctg} A = \frac{\sqrt{1-e^2\cos^2 u}}{\cos u}\,\frac{du}{d\lambda}. \tag{5}$$

If we specify the constant c in the form:

$$c = a\cos u_G \tag{6}$$

and in the environment of the meridian λ_G we limit ourselves to an interval:

$$u \in \langle -u_G, u_G \rangle, \qquad u_G \in (0, \pi/2), \tag{7}$$

in which is satisfied the condition:

$$\frac{du}{d\lambda} < 0,$$

then in interval 7 differential equation 3 will take the form:

$$\frac{\sqrt{1-e^2\cos^2 u}}{\cos u}\,\frac{du}{d\lambda} = \frac{\sqrt{1-\left(\frac{\cos u_G}{\cos u}\right)^2}}{\frac{\cos u_G}{\cos u}}, \tag{8}$$

$$d\lambda = -\frac{\cos u_G}{\cos^2 u}\sqrt{\frac{1-e^2\cos^2 u}{1-\left(\frac{\cos u_G}{\cos u}\right)^2}}\,du; \tag{9}$$

in the case of a sphere $e = 0$, we have:

$$d\lambda = -\frac{\cos u_G\, du}{\cos u\sqrt{\cos^2 u - \cos^2 u_G}}, \tag{10}$$

$$d\lambda = -\frac{\operatorname{ctg} u_G\, du}{\cos^2 u\sqrt{1-\operatorname{ctg}^2 u_G \operatorname{tg}^2 u}}, \tag{11}$$

Substituting:

$$(\operatorname{tg} u = w \operatorname{tg} u_G) \rightarrow \left(\frac{du}{\cos^2 u} = \operatorname{tg} u_G\, dw\right) \tag{12}$$

we obtain:

$$\lambda - \lambda_G = \int \frac{-dw}{\sqrt{1-w^2}} = \arccos w.$$

and thus:

$$(\cos(\lambda-\lambda_G) = \operatorname{ctg} u_G \operatorname{tg} u) \equiv (\operatorname{tg} u = \operatorname{tg} u_G \cos(\lambda-\lambda_G)). \tag{13}$$

the equation system for a geodesic line on a sphere.

This is the equation sought for a geodesic line on a sphere[1].

The line under consideration with equation 13 passes through point $G(u_G, \lambda_G)$ and is orthogonal to meridian λ_G at that point. If parameter u in relation 13 runs across interval 7 once, then parameter $\Delta\lambda = \lambda - \lambda_G$ runs across the interval:

$$\Delta\lambda \in \langle 0, \pi \rangle. \tag{14}$$

The range of variation of parameter $\Delta\lambda$ in interval 7 does not depend on parameter u_G. It is assumed that interval 14 constitutes one half of the period of oscillation, independent of u_G, of the geodesic line (13) in relation to $u = 0$. Let us also see what happens to equation 9 if in its changed form:

$$d\lambda = -\frac{\operatorname{ctg} u_G}{\cos u}\sqrt{\frac{1-e^2+\operatorname{tg}^2 u}{1-\operatorname{ctg}^2 u_G \operatorname{tg}^2 u}}\,du \tag{15}$$

we take the substitution:

$$\left(q = \ln \operatorname{tg}\left(45^\circ + \frac{u}{2}\right)\right) \rightarrow \left(dq = \frac{du}{\cos u}\right). \tag{16}$$

In this we obtain:

$$d\lambda = -\operatorname{ctg} u_G \sqrt{\frac{1-e^2+\sinh^2 q}{1-\operatorname{ctg}^2 u_G \sinh^2 q}}\,dq. \tag{17}$$

Considering equation 17, as well as $e = 1$ and $q > 0$, we have:

$$d\lambda = -\frac{\operatorname{ctg} u_G \sinh q\, dq}{\sqrt{1-\operatorname{ctg}^2 u_G \sinh^2 q}}, \tag{18}$$

$$d\lambda = -\frac{\cos u_G \sinh q\, dq}{\sqrt{1-\cos^2 u_G \cosh^2 q}}. \tag{19}$$

---

[1]In the following interval (7) of revolution of parameter u, i.e., in the interval $\lambda - \lambda_G = \Delta\lambda \in (\pi, 2\pi)$, in which $\frac{du}{d\lambda} > 0$, equation 13 retains its binding force.

Performing further substitution:

$$(\cos u_G \cosh q = w) \to (\cos u_G \sinh q \, dq = dw) \tag{20}$$

we get:

$$\left(d\lambda = \frac{-dw}{\sqrt{1-w^2}}\right) \equiv (\lambda - \lambda_G = \arccos w), \tag{21}$$

$$\cos \Delta\lambda = \cos u_G \cosh\left(\ln \operatorname{tg}\left(45^\circ + \frac{u}{2}\right)\right). \tag{22}$$

Since:

$$\cosh q = \frac{1}{\cos u} \tag{23}$$

relation 22 can finally be written in the form:

$$\cos u_G = \cos u \cos \Delta\lambda. \tag{24}$$

This is an analytic description of segment $\overrightarrow{GG_1}$ of the normal to the semi-axis $\lambda = \lambda_G$, which passes through points:

$$G(u_G, \lambda_G), \qquad G_1(u_{G_1} = 0, \lambda_{G_1} = \lambda_G + u_G). \tag{25}$$

It should be noted that equation 24 at points $u \in (0, -u_G)$ is no longer binding. In passing through zero, parameter u causes a change in the sense of vector $\vec{r}_u$.

The subsequent path of the geodesic line for $e = 1$ in the interval $u \in (0, -u_G)$ with the condition $\frac{du}{d\lambda} < 0$ is predicted by the segment $\overrightarrow{G_1G_2}$ of the normal to semi-axis $\lambda = \lambda_G + 2u_G$. This straight line passes through the points:

$$G_1(u_{G_1} = 0, \lambda_{G_1}^- = \lambda_G + u_G), \qquad G_2(u_{G_2}^- = -u_G, \lambda_{G_2}^- = \lambda_G + 2u_G). \tag{26}$$

On segment $\overrightarrow{G_1G_2}$, with consideration of the characteristic sense of the direction angle A, equation 3 remains satisfied. In this connec-

-tion equation 3 and equation 24, which are related on segment $\overrightarrow{GG_1}$, analytically describe a certain broken line inscribed in a circle. This line passes through point G. The vertices of this broken line depend on the parameter u_G. They are located on the circumference of a circle with radius a at points:

$$\lambda = \lambda_{G_2} + 2ku_G, \quad k = 0, \pm 1, \pm 2, \ldots \tag{27}$$

In the general case $e \in (0, 1)$, by substituting into equation 15:

$$(\cos v = \operatorname{ctg} u_G \operatorname{tg} u) \rightarrow \left(-\sin v \, dv = \frac{\operatorname{ctg} u_G}{\cos^2 u} du\right) \tag{28}$$

we arrive at the equation:

$$d\lambda = \sqrt{\frac{1 - e^2 + \operatorname{tg}^2 u_G \cos^2 v}{1 + \operatorname{tg}^2 u_G \cos^2 v}} \, dv. \tag{29}$$

After making simple transformations, we have:

$$d\lambda = \sqrt{1 - e^2 \cos^2 u_G} \sqrt{\frac{1 - \dfrac{\sin^2 u_G \sin^2 v}{1 - e^2 \cos^2 u_G}}{1 - \sin^2 u_G \sin^2 v}} \, dv. \tag{30}$$

By taking the new variable:

$$\hat{u} = \hat{u}(\sin u_G, v) = \int_0^v \frac{dt}{\sqrt{1 - \sin^2 u_G \sin^2 t}} \tag{31}$$

and integrating equation 30 bilaterally, we finally obtain:

$$\lambda - \lambda_G = \sqrt{1 - e^2 \cos^2 u_G} \int_0^{\hat{u}} \sqrt{1 - \tau^2 \operatorname{sn}^2 t} \, dt, \tag{32}$$

$$\tau = \frac{\sin u_G}{\sqrt{1 - e^2 \cos^2 u_G}}, \tag{33}$$

$$\operatorname{sn} t \stackrel{\text{def}}{\equiv} \sin[\operatorname{am}(\sin u_G, t)]. \tag{34}$$

Here $\operatorname{am}(\sin u_G, t)$ states the inverse function 31 with the substitution $\hat{u} = t$.

The original function 32 is also the function am(τ, $\hat{u}$) and thus the inverse function for the Legendre form elliptic integral of the first kind (31) with parameter 33.

We therefore have:

$$\lambda - \lambda_G = \sqrt{1 - e^2\cos^2 u_G}\, \operatorname{am}(\tau, \hat{u}(k, v)), \tag{35}$$

where

$$k = \sin u_G. \tag{36}$$

The system of relations 28, 31, and 35 presents an interesting dependence between the parameters λ and u.

From 28, 30, and 35 it is evident that 35, as a function of parameter $\Delta\lambda = \lambda - \lambda_G$ is a periodic function with a half-period:

$$\omega = \sqrt{1 - e^2\cos^2 u_G} \int_0^{\pi} \sqrt{\frac{1 - \tau^2\sin^2 v}{1 - k^2\sin^2 v}}\, dv. \tag{37}$$

This half-period depends on u_G and e and always takes values from the interval:

$$\omega \in (0, \pi). \tag{38}$$

This statement is true, because from it and from the assumption that the eccentricity e belongs to the interval (0, 1) it is possible to educe the irreversibility of Soldner's system and thereby to demonstrate the ambiguity of a solution of a so-called inverse problem for a geodesic line.

The path of a geodesic line on a surface (1) has been the subject of studies by many scholars, including H. Schmehl [3], [4], Cayley [1], F. Hopfner [2], and Z. Zorski [5], [6]. References 3 and 4 are among the leading studies in this area. Reference 5 deserves special attention. It definitively explains the problem of ambiguity in the solution of an inverse problem for a geodesic line. In the present study it is shown that the problem of geodesic lines on a

surface (1) has a global solution. From the global solution it is evident that all of the properties known thus far for geodesic lines on a surface (1) are consequences of the combination of the functions which have form 28, 31, and 35.

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Equations of geodesic line on a spheroid

Summary

In the article is presented a certain variant of geodesic line equation for any rotational ellipsoid. In addition to the general case, where excentricity $e \in (0,1)$, two extremities ($e = 0$ and $e = 1$) were separately discussed. It was preofed that if $e = 1$, the geodesic line is a periodical curve of the curvature reversed, inscribed within the circle of radius $= a$, and with the half-period $\omega = 2u_0$

It is shown that in the general case $e \in (0,1)$ the half-period ω depends directly on parameters e and u_0 where u_0 is the reduced latitude of the turn point.

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